



9.3 The matrix inverse

Douglas Wilhelm Harder, LEL, M.Math.

dwharder@uwaterloo.ca

dwharder@gmail.com





Introduction

- In this topic, we will
 - Find an algorithm for finding the inverse of a matrix
 - Look at some examples
 - Consider the numerical stability of finding the inverse





Invertibility

- In secondary school, for a function $f: \mathbf{R} \rightarrow \mathbf{R}$ to be invertible, it had to be both one-to-one and onto
 - If it was not one-to-one, then there would be at least one y value such that there are at least two x_1 and x_2 such $y = f(x_1) = f(x_2)$,
 - Thus, $f^{-1}(y) = x_1$ or $f^{-1}(y) = x_2$?
 - If it was not onto, there would be some y such that there was no x such that $y = f(x)$
 - Thus, $f^{-1}(y)$ is not defined





Invertibility

- Given $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$,
 - That is, $A \in \mathcal{L}(\mathbf{R}^n)$the matrix A is invertible if there exists a matrix A^{-1} such that
$$A^{-1} A = A A^{-1} = I$$
- For a matrix to be invertible, it must be both one-to-one and onto
 - If it was not one-to-one, then $\dim(\text{null}(A)) > 0$,
so the pre-image of $\mathbf{0}$ is not unique
 - If it was not onto, then there would be vectors in $\mathbf{u}$ such that that vector has no pre-image, so $A^{-1}\mathbf{u}$ is not defined
 - Consequently, a matrix is invertible if and only if $\text{rank}(A) = n$





Finding the inverse

- If the rank of A is n , then there must be a sequence of row operations that transforms that matrix into one that is upper triangular

- Each row operation can be written as a matrix R ,
so therefore there exist N_1 row operations such that

$$R_{N_1} R_{N_1-1} R_{N_1-2} \cdots R_3 R_2 R_1 A = U$$

- Now, if the rank(A) = n , then the diagonal entries of U are all non-zero, so they are all invertible as elements in their field
- Thus, there exist another n row operations that make all the diagonal entries 1

$$R_{u_{n,n}^{-1}Rn}^{-1} R_{u_{n-1,n-1}^{-1}Rn-1}^{-1} \cdots R_{u_{2,2}^{-1}R2}^{-1} R_{u_{1,1}^{-1}R1}^{-1} R_{N_1} R_{N_1-1} R_{N_1-2} \cdots R_3 R_2 R_1 A = U'$$





Finding the inverse

- Now, suppose you have a square upper-triangular matrix U' where all the entries on the diagonal are now 1:

$$\begin{pmatrix} 1 & u'_{1,2} & u'_{1,3} & \cdots & u'_{1,n} \\ 0 & 1 & u'_{2,3} & \cdots & u'_{2,n} \\ 0 & 0 & 1 & \cdots & u'_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$





Finding the inverse

- Could we not now apply another sequence of row operations to find this matrix is row-equivalent to the identity matrix?

$$\begin{pmatrix} 1 & u'_{1,2} & u'_{1,3} & \cdots & u'_{1,n} \\ 0 & 1 & u'_{2,3} & \cdots & u'_{2,n} \\ 0 & 0 & 1 & \cdots & u'_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- You would start with the row operation

$$R_{-u'_{n-1,n}Rn \rightarrow Rn-1}$$

which add a multiple of Row n onto Row $n - 1$ to zero out the entry at index $(n - 1, n)$





Finding the inverse

- Thus, the application of these row operations is nothing more than multiplying on the left by the appropriate matrix, so if

$$R_{u_{n,n}^{-1}Rn} R_{u_{n-1,n-1}^{-1}Rn-1} \cdots R_{u_{2,2}^{-1}R2} R_{u_{1,1}^{-1}R1} R_{N_1} R_{N_1-1} R_{N_1-2} \cdots R_3 R_2 R_1 A = U'$$

then there are N_2 additional row operations such that

$$R'_{N_2} R'_{N_2-1} \cdots R'_2 R'_1 R_{u_{n,n}^{-1}Rn} R_{u_{n-1,n-1}^{-1}Rn-1} \cdots R_{u_{2,2}^{-1}R2} R_{u_{1,1}^{-1}R1} R_{N_1} R_{N_1-1} \cdots R_2 R_1 A = I$$

- In this case, we have that

$$\left(R'_{N_2} R'_{N_2-1} \cdots R'_2 R'_1 R_{u_{n,n}^{-1}Rn} R_{u_{n-1,n-1}^{-1}Rn-1} \cdots R_{u_{2,2}^{-1}R2} R_{u_{1,1}^{-1}R1} R_{N_1} R_{N_1-1} \cdots R_2 R_1 \right) A = I$$

and the product of square matrices is a matrix, so

$$A^{-1} = R'_{N_2} R'_{N_2-1} \cdots R'_2 R'_1 R_{u_{n,n}^{-1}Rn} R_{u_{n-1,n-1}^{-1}Rn-1} \cdots R_{u_{2,2}^{-1}R2} R_{u_{1,1}^{-1}R1} R_{N_1} R_{N_1-1} \cdots R_2 R_1$$





Finding the inverse

- Thus, all we must do is find accumulate the matrices in question
 - Strategy: start with the matrix augmented with the identity

$$\left(\begin{array}{cc|cc} 2.0 & -0.8 & 1 & 0 \\ 1.0 & -4.4 & 0 & 1 \end{array} \right)$$

- Recall that applying a row operation R_1 is the same as multiplying on the left by the corresponding matrix

$$R_1 \left(\begin{array}{cc|cc} 2.0 & -0.8 & 1 & 0 \\ 1.0 & -4.4 & 0 & 1 \end{array} \right)$$

- Recall, however, that multiplying on the left by a matrix is the same as multiplying each column by R_1

$$R_1 \left(\begin{array}{cc|cc} 2.0 & -0.8 & 1 & 0 \\ 1.0 & -4.4 & 0 & 1 \end{array} \right) = \left(R_1 \left(\begin{array}{c} 2.0 \\ 1.0 \end{array} \right) \quad R_1 \left(\begin{array}{c} -0.8 \\ -4.4 \end{array} \right) \mid R_1 \left(\begin{array}{c} 1 \\ 0 \end{array} \right) \quad R_1 \left(\begin{array}{c} 0 \\ 1 \end{array} \right) \right)$$





Finding the inverse

- Now, we have

$$\begin{aligned} R_1 \left(\begin{array}{cc|cc} 2.0 & -0.8 & 1 & 0 \\ 1.0 & -4.4 & 0 & 1 \end{array} \right) &= \left(R_1 \left(\begin{array}{c} 2.0 \\ 1.0 \end{array} \right) \quad R_1 \left(\begin{array}{c} -0.8 \\ -4.4 \end{array} \right) \mid R_1 \left(\begin{array}{c} 1 \\ 0 \end{array} \right) \quad R_1 \left(\begin{array}{c} 0 \\ 1 \end{array} \right) \right) \\ &= \left(R_1 \left(\begin{array}{cc} 2.0 & -0.8 \\ 1.0 & -4.4 \end{array} \right) \mid R_1 \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \right) \\ &= \left(R_1 \left(\begin{array}{cc} 2.0 & -0.8 \\ 1.0 & -4.4 \end{array} \right) \mid R_1 \right) \end{aligned}$$





Finding the inverse

- Now, applying a second row operation, we have

$$R_2R_1\left(\begin{array}{cc|cc} 2.0 & -0.8 & 1 & 0 \\ 1.0 & -4.4 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|cc} R_2R_1\left(\begin{array}{cc} 2.0 & -0.8 \end{array}\right) \\ R_2R_1\left(\begin{array}{cc} 1.0 & -4.4 \end{array}\right) \end{array}\right)$$

- If we now applied all the necessary row operations to convert the matrix A to the identity matrix I , then the right-hand matrix stores all the corresponding row operations





Finding the inverse

- For example,

$$\begin{aligned} \left(\begin{array}{cc|cc} 2.0 & -0.8 & 1 & 0 \\ 1.0 & -4.4 & 0 & 1 \end{array} \right) &\sim \left(\begin{array}{cc|cc} 2.0 & -0.8 & 1 & 0 \\ 0 & -4.0 & -0.5 & 1 \end{array} \right) \\ &\sim \left(\begin{array}{cc|cc} 1 & -0.4 & 0.5 & 0 \\ 0 & -4.0 & -0.5 & 1 \end{array} \right) \\ &\sim \left(\begin{array}{cc|cc} 1 & -0.4 & 0.5 & 0 \\ 0 & 1 & 0.125 & -0.25 \end{array} \right) \\ &\sim \left(\begin{array}{cc|cc} 1 & 0 & 0.55 & -0.1 \\ 0 & 1 & 0.125 & -0.25 \end{array} \right) \end{aligned}$$

- Thus, $\begin{pmatrix} 2.0 & -0.8 \\ 1.0 & -4.4 \end{pmatrix}^{-1} = \begin{pmatrix} 0.55 & -0.1 \\ 0.125 & -0.25 \end{pmatrix}$





Finding the inverse

- For example,

$$\begin{aligned} \left(\begin{array}{cc|cc} 5.0 & -3.5 & 1 & 0 \\ 1.5 & -3.05 & 0 & 1 \end{array} \right) &\sim \left(\begin{array}{cc|cc} 5.0 & -3.5 & 1 & 0 \\ 0 & -2.0 & -0.3 & 1 \end{array} \right) \\ &\sim \left(\begin{array}{cc|cc} 1 & -0.7 & 0.2 & 0 \\ 0 & -2.0 & -0.3 & 1 \end{array} \right) \\ &\sim \left(\begin{array}{cc|cc} 1 & -0.7 & 0.2 & 0 \\ 0 & 1 & 0.15 & -0.5 \end{array} \right) \\ &\sim \left(\begin{array}{cc|cc} 1 & 0 & 0.305 & -0.35 \\ 0 & 1 & 0.15 & -0.5 \end{array} \right) \end{aligned}$$

- Thus, $\begin{pmatrix} 5.0 & -3.5 \\ 1.5 & -3.05 \end{pmatrix}^{-1} = \begin{pmatrix} 0.305 & -0.35 \\ 0.15 & -0.5 \end{pmatrix}$





Finding the inverse

- For example,

$$\begin{pmatrix} -5 & -6 & -2 & | & 1 & 0 & 0 \\ 8 & 9 & 3 & | & 0 & 1 & 0 \\ -4 & -5 & -2 & | & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 8 & 9 & 3 & | & 0 & 1 & 0 \\ 0 & -0.375 & -0.125 & | & 1 & 0.625 & 0 \\ 0 & -0.5 & -0.5 & | & 0 & 0.5 & 1 \end{pmatrix}$$
$$\sim \begin{pmatrix} 8 & 9 & 3 & | & 0 & 1 & 0 \\ 0 & -0.5 & -0.5 & | & 0 & 0.5 & 1 \\ 0 & 0 & 0.25 & | & 1 & 0.25 & -0.75 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 1.125 & 0.375 & | & 0 & 0.125 & 0 \\ 0 & 1 & 1 & | & 0 & -1 & -2 \\ 0 & 0 & 1 & | & 4 & 1 & -3 \end{pmatrix}$$





Finding the inverse

- Thus, so far, we have:

$$\begin{pmatrix} -5 & -6 & -2 & | & 1 & 0 & 0 \\ 8 & 9 & 3 & | & 0 & 1 & 0 \\ -4 & -5 & -2 & | & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1.125 & 0.375 & | & 0 & 0.125 & 0 \\ 0 & 1 & 1 & | & 0 & -1 & -2 \\ 0 & 0 & 1 & | & 4 & 1 & -3 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 1.125 & 0 & | & -1.5 & -0.25 & 1.125 \\ 0 & 1 & 0 & | & -4 & -2 & 1 \\ 0 & 0 & 1 & | & 4 & 1 & -3 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 0 & | & 3 & 2 & 0 \\ 0 & 1 & 0 & | & -4 & -2 & 1 \\ 0 & 0 & 1 & | & 4 & 1 & -3 \end{pmatrix}$$

- Thus, $\begin{pmatrix} -5 & -6 & -2 \\ 8 & 9 & 3 \\ -4 & -5 & -2 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 2 & 0 \\ -4 & -2 & 1 \\ 4 & 1 & -3 \end{pmatrix}$





Numerical stability

- Recall that in order to make Gaussian elimination numerically stable, it was required that we are able to apply partial pivoting:
 - For invertible matrices,
this is equivalent to putting the largest entry on the diagonal
- Thus, it is likely that some of these row operations are row swaps to allow for partial pivoting:

$$R_{N_1} R_{N_1-1} R_{N_1-2} \cdots R_3 R_2 R_1 A = U$$





Numerical stability

- Question: If we've gotten to this point:

$$\begin{pmatrix} 1 & u'_{1,2} & u'_{1,3} & \cdots & u'_{1,n} \\ 0 & 1 & u'_{2,3} & \cdots & u'_{2,n} \\ 0 & 0 & 1 & \cdots & u'_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- Can we perform any row swaps?
- A row swap would no longer leave the diagonal entries all one, so no...





Numerical stability

- Consequence:
Finding the inverse of a matrix is not numerically stable
- Thus, in some applications of linear algebra to vector calculus, finding the inverse is appropriate
 - This is because the matrices involved tend to be amenable to this algorithm
- However, in general, if you want to solve $A\mathbf{u}_1 = \mathbf{u}_2$,
 - To get a good approximation of the formula, you must apply Gaussian elimination with partial pivoting and backward substitution
 - Never calculate A^{-1} and then find $\mathbf{u}_1 = A^{-1}\mathbf{u}_2$





Numerical stability

- For example, in the third example, we got to the augmented matrix

$$\left(\begin{array}{ccc|ccc} 1 & 1.125 & 0 & -1.5 & -0.25 & 1.125 \\ 0 & 1 & 0 & -4 & -2 & 1 \\ 0 & 0 & 1 & 4 & 1 & -3 \end{array} \right)$$

- At this point, we had to add -1.125 times Row 2 onto Row 1
 - There is nothing we could do to avoid this in finding the inverse





Finding the inverse

- For example,

$$\begin{pmatrix} 0.5 & 2.5 & -2 & | & 1 & 0 & 0 \\ 0.2 & 0.75 & -0.05 & | & 0 & 1 & 0 \\ -0.35 & -1.825 & 1.825 & | & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0.5 & 2.5 & -2 & | & 1 & 0 & 0 \\ 0 & -0.25 & 0.75 & | & -0.4 & 1 & 0 \\ 0 & -0.075 & 0.425 & | & 0.7 & 0 & 1 \end{pmatrix}$$
$$\sim \begin{pmatrix} 0.5 & 2.5 & -2 & | & 1 & 0 & 0 \\ 0 & -0.25 & 0.75 & | & -0.4 & 1 & 0 \\ 0 & 0 & 0.2 & | & 0.82 & -0.3 & 1 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 5 & -4 & | & 2 & 0 & 0 \\ 0 & 1 & -3 & | & 1.6 & -4 & 0 \\ 0 & 0 & 1 & | & 4.1 & -1.5 & 5 \end{pmatrix}$$





Finding the inverse

- Thus, so far, we have:

$$\begin{pmatrix} 0.5 & 2.5 & -2 & | & 1 & 0 & 0 \\ 0.2 & 0.75 & -0.05 & | & 0 & 1 & 0 \\ -0.35 & -1.825 & 1.825 & | & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 5 & -4 & | & 2 & 0 & 0 \\ 0 & 1 & -3 & | & 1.6 & -4 & 0 \\ 0 & 0 & 1 & | & 4.1 & -1.5 & 5 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 5 & 0 & | & 18.4 & -6 & 20 \\ 0 & 1 & 0 & | & 13.9 & -8.5 & 15 \\ 0 & 0 & 1 & | & 4.1 & -1.5 & 5 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 0 & | & -51.1 & 36.5 & -55 \\ 0 & 1 & 0 & | & 13.9 & -8.5 & 15 \\ 0 & 0 & 1 & | & 4.1 & -1.5 & 5 \end{pmatrix}$$

- Thus, $\begin{pmatrix} 0.5 & 2.5 & -2 \\ 0.2 & 0.75 & -0.05 \\ -0.35 & -1.825 & 1.825 \end{pmatrix}^{-1} = \begin{pmatrix} -51.1 & 36.5 & -55 \\ 13.9 & -8.5 & 15 \\ 4.1 & -1.5 & 5 \end{pmatrix}$





Summary

- Following this topic, you now
 - Know an algorithm for finding the inverse of a matrix
 - Have seen the application of this algorithm
 - Understand that finding this inverse is numerically unstable
 - Finding the inverse should be restricted to applications such as vector calculus where it is clearer the inverse is stable





References

- [1] https://en.wikipedia.org/wiki/Invertible_matrix





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

<https://www.rbg.ca/>

for more information.





Disclaimer

These slides are provided for the NE 112 *Linear algebra for nanotechnology engineering* course taught at the University of Waterloo. The material in it reflects the authors' best judgment in light of the information available to them at the time of preparation. Any reliance on these course slides by any party for any other purpose are the responsibility of such parties. The authors accept no responsibility for damages, if any, suffered by any party as a result of decisions made or actions based on these course slides for any other purpose than that for which it was intended.

